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THE PROBLEM OF AN INCLUSION IN A THREE-DIMENSIONAL ELASTIC WEDGE†

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Fundamental solutions for a three-dimensional wedge are used to investigate problems of a thin, rigid, elliptic inclusion in a wedge. A regular asymptotic form is employed which has previously been used in contact problems for a wedge [1] and in problems of a crack in a wedge [2] in the case of an elliptic shape of the contact region or crack. The method is effective in the case of an inclusion which is sufficiently distant from an edge of the wedge when the known exact solution for the space [3] can be taken as the zeroth approximation. A numerical analysis and comparison of different characteristics of wedge problems is carried out.

Fredholm integral equations of the second kind were obtained [4], in terms of the solution of which the displacements and stresses in a three-dimensional elastic wedge, acted upon by normal and shear loads on one of its edges and different conditions on the other edge, were expressed. For the case, when this edge is stress-free, Papkovich–Neuber functions have been presented in [5] which are identical to the well-known solutions of the Boussinesq and Cerruti problems when the angle of the wedge is equal to π (the case of a half-space). A complex Fourier–Kontorovich–Lebedev integral was used to construct the solutions in [4, 5] and also the technique of reducing the three-dimensional problem of the theory of elasticity to a Vekua generalized Hilbert boundary-value problem [6, 7].

The exact solutions of two boundary-value problems are obtained below using this technique when an arbitrarily directed concentrated force acts in the bisectorial half-plane of the wedge and the faces of the wedge are under conditions of sliding clamping (Problem A) and rigid clamping (Problem B). When the aperture angle of the wedge is equal to 2π, the solution of Problem A is identical with Kelvin's fundamental solution [8] in the classical theory of elasticity. Problem A generalizes the mixed problem for a wedge (the normal displacements and shear stresses on the faces are specified) [9] and Problem B generalizes the second basic problem for a wedge [10] to the case of the action of forces inside the wedge.

1. A CONCENTRATED FORCE INSIDE A THREE-DIMENSIONAL WEDGE

Consider a three-dimensional elastic wedge ($0 \leqslant r < \infty$, $|\varphi| \leqslant \alpha$, $|z| < \infty$) with aperture angle 2α and elastic characteristics G (the shear modulus) and ν (Poisson's ratio) in cylindrical coordinates r, φ and z. The z axis is directed along an edge of the wedge such that the system of coordinates is a right-handed system (Fig. 1). Suppose an arbitrary concentrated force $\mathbf{P}$, which has the projections P_r and P_z on the coordinates axes, acts at the point $r = x$, $z = y$ in the middle half-plane $\varphi = 0$ of the wedge. The faces $\varphi = \pm\alpha$ are under conditions of sliding or rigid clamping (Problems A and B, respectively). By virtue of the symmetry of the problem with respect to the coordinate φ, we shall consider the domain of the wedge $-\alpha \leqslant \varphi \leqslant 0$ and write the boundary conditions in the form

$$
\begin{aligned}
&\varphi=-\alpha: \quad u_\varphi=\tau_{r\varphi}=\tau_{\varphi z}=0 \quad \text{Problem A}\\
&\varphi=-\alpha: \quad u_\varphi=u_r=u_z=0 \quad \text{Problem B}\\
&\varphi=0: \quad u_\varphi=0, \quad \tau_{r\varphi}=\frac{1}{2}P_r\delta(r-x)\delta(z-y), \quad \tau_{\varphi z}=\frac{1}{2}P_z\delta(r-x)\delta(z-y)
\end{aligned} \tag{1.1}
$$

It is also assumed that the stresses decrease at infinity.

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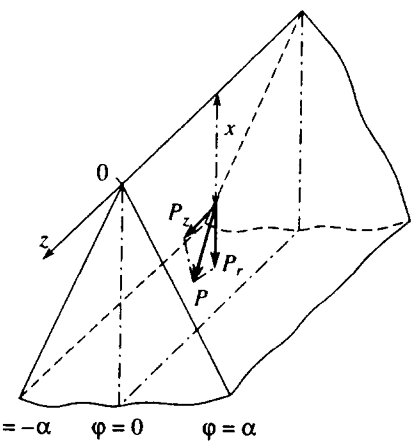

Fig. 1

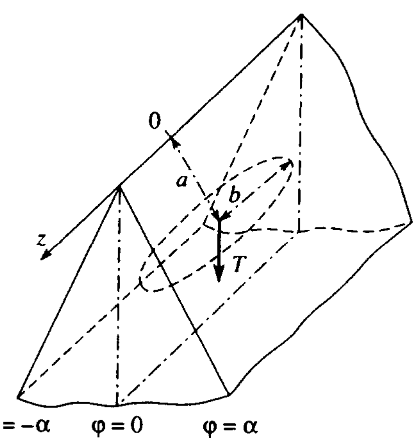

Fig. 2

We shall express the general solution of the Navier equilibrium equations in cylindrical coordinates in terms of three Papkovich–Neuber harmonic functions $\Phi_n = \Phi_n(r, \varphi, z)$ $(n = 0, 1, 2)$ using the formulae

$$
\begin{aligned}
u_r &= \frac{\partial \Phi_0}{\partial r} + \frac{1}{4(1-\nu)} \frac{\partial}{\partial r}(r\omega_1) - \omega_1, \quad \omega_1 = \sin\varphi\Phi_1 - \cos\varphi\Phi_2 \\
u_\varphi &= \frac{1}{r}\frac{\partial \Phi_0}{\partial \varphi} + \frac{1}{4(1-\nu)} \frac{\partial \omega_1}{\partial \varphi} - \omega_2, \quad \omega_2 = \cos\varphi\Phi_1 + \sin\varphi\Phi_2 \\
u_z &= \frac{\partial \Phi_0}{\partial z} + \frac{r}{4(1-\nu)} \frac{\partial \omega_1}{\partial z}
\end{aligned}
\tag{1.2}
$$

Hence the stresses can be determined using Hooke's law.

We shall seek the harmonic functions Φ_n in the form of Fourier integrals with respect to z and Kontorovich–Lebedev integrals with respect to r. Using the well known technique [6, 7], we find the solution of the boundary-value problems (1.1) in the form of (1.2), where

$$
\begin{aligned}
\Phi_n(r,\varphi,z) &= \frac{1}{\pi^3 G} \int_0^\infty \int_0^\infty \operatorname{sh}(\pi\tau)\mathbf{K}_{i\tau}(\beta r)\{P_r C_n^+(\tau,\beta)\cos(\beta[z-y]) + \\
&+ P_z C_n^-(\tau,\beta)\sin(\beta[z-y])\beta^{-1}\} d\tau d\beta, \quad n = 0,1,2
\end{aligned}
\tag{1.3}
$$

Here $\mathbf{K}_{i\tau}(x)$ is the modified Bessel function. The functions

$$
\begin{aligned}
&C_n^{\pm} = A_n^{\pm}(\tau,\beta)\operatorname{ch}(\varphi\tau) + B_n^{\pm}(\tau,\beta)\operatorname{sh}(\varphi\tau)\\
&B_0^{+}(\tau,\beta) = \frac{x}{4(1-\nu)}\mathbf{K}_{i\tau}(\beta x),\quad A_1^{+}(\tau,\beta)=0,\quad B_2^{+}(\tau,\beta) = \mathbf{K}_{i\tau}(\beta x)\\
&B_0^{-}(\tau,\beta) = \mathbf{K}_{i\tau}(\beta x) + \frac{x}{4(1-\nu)}\mathbf{K}'_{i\tau}(\beta x),\quad A_1^{-}(\tau,\beta) = \frac{\tau}{x}\mathbf{K}_{i\tau}(\beta x)\\
&B_2^{-}(\tau,\beta) = \mathbf{K}'_{i\tau}(\beta x) = \frac{\partial}{\partial x}\mathbf{K}_{i\tau}(\beta x)
\end{aligned}
\tag{1.4}
$$

are the same for both of the problems being considered. Then, we have

$$
\begin{aligned}
&A_0^{+}(\tau,\beta) = \frac{x\operatorname{cth}(\alpha\tau)}{4(1-\nu)}\mathbf{K}_{i\tau}(\beta x),\quad B_1^{+}(\tau,\beta) = -\frac{\sin(2\alpha)}{g_-(\tau,\alpha)}\mathbf{K}_{i\tau}(\beta x),\quad A_2^{+}(\tau,\beta) = \frac{\operatorname{sh}(2\alpha\tau)}{g_-(\tau,\alpha)}\mathbf{K}_{i\tau}(\beta x)\\
&A_0^{-}(\tau,\beta) = \operatorname{cth}(\alpha\tau)\left[\mathbf{K}_{i\tau}(\beta x) + \frac{x}{\varkappa+1}\mathbf{K}'_{i\tau}(\beta x)\right]\\
&B_1^{-}(\tau,\beta) = \frac{\tau\operatorname{sh}(2\alpha\tau)\mathbf{K}_{i\tau}(\beta x) - x\sin(2\alpha)\mathbf{K}'_{i\tau}(\beta x)}{xg_-(\tau,\alpha)}\\
&A_2^{-}(\tau,\beta) = \frac{\tau\sin(2\alpha)\mathbf{K}_{i\tau}(\beta x) + x\operatorname{sh}(2\alpha\tau)\mathbf{K}'_{i\tau}(\beta x)}{xg_-(\tau,\alpha)}
\end{aligned}
\tag{1.5}
$$

in the case of Problem A and

$$
\begin{aligned}
&A_0^{+}(\tau,\beta) = \frac{x\operatorname{th}(\alpha\tau)}{\varkappa+1}\mathbf{K}_{i\tau}(\beta x)\\
&B_1^{+}(\tau,\beta) = \sin(2\alpha)\frac{x\operatorname{sh}(2\alpha\tau)\mathbf{K}'_{i\tau}(\beta x) - \tau\sin(2\alpha)\mathbf{K}_{i\tau}(\beta x)}{g_+(\tau,\alpha)g(\tau,\alpha)}\\
&A_2^{+}(\tau,\beta) = 2\operatorname{sh}^2(\alpha\tau)\frac{[\varkappa g_+(\tau,\alpha) - \tau\operatorname{cth}(\alpha\tau)\sin(2\alpha)]\mathbf{K}_{i\tau}(\beta x) + 2x\sin^2\alpha\mathbf{K}'_{i\tau}(\beta x)}{g_+(\tau,\alpha)g(\tau,\alpha)}\\
&A_0^{-}(\tau,\beta) = \operatorname{th}(\alpha\tau)\left[\mathbf{K}_{i\tau}(\beta x) + \frac{x}{\varkappa+1}\mathbf{K}'_{i\tau}(\beta x)\right]\\
&B_1^{-}(\tau,\beta) = \frac{\sin(2\alpha)}{g_+(\tau,\alpha)}\mathbf{K}'_{i\tau}(\beta x) + \left[\frac{\beta^2 x\sin(2\alpha)\operatorname{sh}(2\alpha\tau)}{g_+(\tau,\alpha)g(\tau,\alpha)} + \frac{\tau\operatorname{sh}(2\alpha\tau)}{xg_+(\tau,\alpha)}\right]\mathbf{K}_{i\tau}(\beta x)\\
&A_2^{-}(\tau,\beta) = \frac{\operatorname{sh}(2\alpha\tau)}{g_+(\tau,\alpha)}\mathbf{K}'_{i\tau}(\beta x) + \left[\frac{4\beta^2 x\sin^2\alpha\operatorname{sh}^2(\alpha\tau)}{g_+(\tau,\alpha)g(\tau,\alpha)} - \frac{\tau\sin(2\alpha)}{xg_+(\tau,\alpha)}\right]\mathbf{K}_{i\tau}(\beta x)
\end{aligned}
\tag{1.6}
$$

in the case of Problem B.

Here

$$
g_{\pm}(\tau,\alpha) = \operatorname{ch}(2\alpha\tau) \pm \cos(2\alpha),\quad g(\tau,\alpha) = \varkappa\operatorname{sh}(2\alpha\tau) - \tau\sin(2\alpha),\quad \varkappa = 3-4\nu
$$

In the case of the functions (1.4)–(1.6), the integrals (1.3) converge for all $\varphi \in [-\alpha, 0]$.

The solution of Problem A in the form of (1.3)–(1.5) when $\alpha = \pi$ is identical to the fundamental Kelvin solution for an elastic half-space. Actually, in this case

$$A_0^+(\tau,\beta)=\frac{x\,\mathrm{cth}(\pi\tau)}{\varkappa+1}\mathbf{K}_{i\tau}(\beta x),\quad B_1^+(\tau,\beta)=0,\quad A_2^+(\tau,\beta)=\mathrm{cth}(\pi\tau)\mathbf{K}_{i\tau}(\beta x)$$

$$A_0^-(\tau,\beta)=\mathrm{cth}(\pi\tau)\left[\mathbf{K}_{i\tau}(\beta x)+\frac{x}{\varkappa+1}\mathbf{K}'_{i\tau}(\beta x)\right] \tag{1.7}$$

$$B_1^-(\tau,\beta)=\frac{\tau}{x}\mathrm{cth}(\pi\tau)\mathbf{K}_{i\tau}(\beta x),\quad A_2^-(\tau,\beta)=\mathrm{cth}(\pi\tau)\mathbf{K}'_{i\tau}(\beta x)$$

and, for example, for the displacements in the plane of action of the force **P**, we obtain, using relations (1.2) and well-known formulae ([11], formulae 2.16.52.6, 2.16.14.1), the expressions

$$u_r(r,0,z)=\frac{P_rR_{11}+P_zR_{12}}{4\pi G},\quad u_z(r,0,z)=\frac{P_rR_{21}+P_zR_{22}}{4\pi G} \tag{1.8}$$

where

$$R_{11}=\frac{1}{R}-\frac{(z-y)^2}{(\varkappa+1)R^3},\quad R_{12}=R_{21}=\frac{(r-x)(z-y)}{(\varkappa+1)R^3}$$
$$R_{22}=\frac{1}{R}-\frac{(r-x)^2}{(\varkappa+1)R^3},\quad R=[(r-x)^2+(z-y)^2]^{1/2} \tag{1.9}$$

They correspond exactly to the fundamental Kelvin solution ([12], formulae (9.2) and (9.4)).

We will now explain why Problems A and B have exact solutions. It is well known [4, 7] that problems of the action of a concentrated force on one face of a three-dimensional wedge, the other face of which is under conditions of sliding or rigid clamping, reduce to Fredholm integral equations of the second kind. The displacements in the wedge can then be expressed in the form of Neuman operator series. The boundary conditions (1.1) considered above correspond to inverse problems since, when $\varphi = 0$, the displacement u_φ is specified instead of the stress σ_φ. Consequently, the solutions of problem (1.1) must contain the inverse of the above-mentioned Neuman series in the form ([2], formula (1.6))

$$\left[\sum_{n=0}^{\infty}(1-2\nu)^nT^n\right]^{-1}=I-(1-2\nu)T \tag{1.10}$$

where T is a known operator and I is the identity operator.

2. AN ELLIPTIC INCLUSION INSIDE A THREE-DIMENSIONAL WEDGE

We will now apply the formula obtained above to problems of a thin, rigid inclusion in the middle half-plane $\varphi = 0$ of a three-dimensional wedge. Suppose this inclusion occupies an elliptic domain Ω: $(r-a)^2/c^2 + z^2/b^2 \leqslant 1$, $a > c$, $b \geqslant c$ (Fig. 2). There is complete adhesion between the inclusion and the elastic medium in the contact region. For simplicity, we will assume that the force T, which is applied to the inclusion and acts in the half-plane $\varphi = 0$, is perpendicular to the edge of the wedge. Consequently, the inclusion is moved by an amount δ in the direction of action of the force. The faces of the wedge are under conditions of sliding or rigid clamping (the Problems of inclusion A and B, respectively). It is required to determine the shear contact stresses $\tau_{r\varphi}(r, 0, z) = 2\tau_1(r, z)$ and $\tau_{\varphi z}(r, 0, z) = 2\tau_2(r, z)$, $(r, z) \in \Omega$ and the relation between the quantities T and δ (only one of these quantities is specified).

Since the problems are symmetrical about the half-plane $\varphi = 0$, we will consider the problems of the equilibrium of an elastic wedge $-\alpha \leqslant \varphi \leqslant 0$ with boundary conditions (1.1) for Problems A and B when $\varphi = -\alpha$ and the following boundary conditions

$$\varphi=0:\quad u_r=\delta,\quad u_z=0\quad (r,z)\in\Omega;\quad \tau_{r\varphi}=0,\quad \tau_{\varphi z}=0\quad (r,z)\notin\Omega;\quad u_\varphi=0 \tag{2.1}$$

Using the solutions obtained, we express the displacements $u_r(r, 0, z)$ and $u_z(r, 0, z)$ using formulae (1.2). On replacing P_r by $2\tau_1(x, y)$ and P_z by $2\tau_2(x, y)$ in these expressions, integrating over the domain Ω with respect to the variables x and y and satisfying the first two conditions of (2.1), we obtain a system of two integral equations in the unknown contact stresses $\tau_n(x,y)$ $(n = 1, 2)$, $(x, y) \in \Omega$. On introducing the dimensionless quantities

$$r_* = \frac{r-a}{b}, \quad x_* = \frac{x-a}{b}, \quad z_* = \frac{z}{b}, \quad y_* = \frac{y}{b}, \quad \delta_* = \frac{\delta}{b}, \quad c_* = \frac{c}{b} \tag{2.2}$$
$$\lambda = \frac{a}{b}, \quad \tau_n^*(r_*, z_*) = \frac{\tau_n(r,z)}{G}, \quad n = 1,2, \quad \Omega_* : \frac{r_*^2}{c_*^2} + z_*^2 \leqslant 1, \quad T_* = \frac{T}{Gb^2}$$

and henceforth omitting the asterisks, this system can be written in the form

$$\begin{gathered}\int_\Omega \tau_1(x,y)\left[\frac{1}{R} - \frac{(z-y)^2}{4(1-\nu)R^3} + K_{11}(x,y,r,z)\right]d\Omega_{xy} + \\ +\int_\Omega \tau_2(x,y)\left[\frac{(r-x)(z-y)}{4(1-\nu)R^3} + K_{12}(x,y,r,z)\right]d\Omega_{xy} = 2\pi\delta \quad (r,z) \in \Omega_{rz} \\ \int_\Omega \tau_1(x,y)\left[\frac{(r-x)(z-y)}{4(1-\nu)R^3} + K_{21}(x,y,r,z)\right]d\Omega_{xy} + \\ +\int_\Omega \tau_2(x,y)\left[\frac{1}{R} - \frac{(r-x)^2}{4(1-\nu)R^3} + K_{22}(x,y,r,z)\right]d\Omega_{xy} = 0 \quad (r,z) \in \Omega_{rz}\end{gathered} \tag{2.3}$$

where

$$\begin{aligned}
&K_{11}(x,y,r,z) = \frac{1}{(1-\nu)\pi^2}\int_0^\infty\int_0^\infty \mathrm{sh}(\pi u)\cos(\beta[z-y])\left\{W_1(u,\alpha)(\lambda+x)\frac{\partial}{\partial r} + W_2(u,\alpha)D_r + \right.\\
&\left. + W_3(u,\alpha)(\lambda+x)\frac{\partial}{\partial x}D_x\right\}\mathbf{K}_{iu}(\beta[\lambda+x])\mathbf{K}_{iu}(\beta[\lambda+r])du d\beta \\
&K_{12}(x,y,r,z) = \frac{1}{(1-\nu)\pi^2}\int_0^\infty\int_0^\infty \mathrm{sh}(\pi u)\frac{\sin(\beta[z-y])}{\beta}\left\{W_1(u,\alpha)\frac{\partial}{\partial r}D_x + \right.\\
&\left. + D_r W(u,\alpha,x)\right\}\mathbf{K}_{iu}(\beta[\lambda+x])\mathbf{K}_{iu}(\beta[\lambda+r])du d\beta \\
&K_{21}(x,y,r,z) = -\frac{\lambda+x}{(1-\nu)\pi^2}\int_0^\infty\int_0^\infty \mathrm{sh}(\pi u)\beta\sin(\beta[z-y])\left\{W_1(u,\alpha) - W_2(u,\alpha)\frac{\lambda+r}{\lambda+x} - \right.\\
&\left. - W_3(u,\alpha)(\lambda+r)\frac{\partial}{\partial x}\right\}\mathbf{K}_{iu}(\beta[\lambda+x])\mathbf{K}_{iu}(\beta[\lambda+r])du d\beta \\
&K_{22}(x,y,r,z) = \frac{1}{(1-\nu)\pi^2}\int_0^\infty\int_0^\infty \mathrm{sh}(\pi u)\cos(\beta[z-y])\{W_1(u,\alpha)D_x - \\
&-(\lambda+r)W(u,\alpha,x)\}\mathbf{K}_{iu}(\beta[\lambda+x])\mathbf{K}_{iu}(\beta[\lambda+r])du d\beta \\
&D_r = \varkappa - (\lambda+r)\frac{\partial}{\partial r}, \quad D_x = \varkappa + 1 + (\lambda+x)\frac{\partial}{\partial x} \\
&W(u,\alpha,x) = W_3(u,\alpha)\beta^2(\lambda+x) + W_4(u,\alpha)\frac{\partial}{\partial x} + \frac{1}{\lambda+x}W_5(u,\alpha)
\end{aligned} \tag{2.4}$$

For Problem A, we have

$$\begin{gathered}W_1(u,\alpha) = \mathrm{cth}(\alpha u) - \mathrm{cth}(\pi u), \quad W_2(u,\alpha) = \frac{\mathrm{sh}(2\alpha u)}{g_-(u,\alpha)} - \mathrm{cth}(\pi u) \\ W_3(u,\alpha) \equiv 0, \quad W_4(u,\alpha) \equiv W_2(u,\alpha), \quad W_5(u,\alpha) = \frac{u\sin(2\alpha)}{g_-(u,\alpha)}\end{gathered} \tag{2.5}$$

and, for Problem B

$$
\begin{aligned}
&W_1(u,\alpha) = \mathrm{th}(\alpha u) - \mathrm{cth}(\pi u) \\
&W_2(u,\alpha) = 2\,\mathrm{sh}^2(\alpha u)\frac{\varkappa g_+(u,\alpha) - u\,\mathrm{cth}(\alpha u)\sin(2\alpha)}{g_+(u,\alpha)g(u,\alpha)} - \mathrm{cth}(\pi u) \\
&W_3(u,\alpha) = \frac{4\sin^2\alpha\,\mathrm{sh}^2(\alpha u)}{g_+(u,\alpha)g(u,\alpha)} \\
&W_4(u,\alpha) = \frac{\mathrm{sh}(2\alpha u)}{g_+(u,\alpha)} - \mathrm{cth}(\pi u), \quad W_5(u,\alpha) = -\frac{u^2\sin(2\alpha)}{g_+(u,\alpha)}
\end{aligned}
\tag{2.6}
$$

Note that $W_m(\alpha, u) = O(\exp[-2\alpha u])$, $u \to \infty$ $(m = 1, \ldots, 5)$ for a fixed value of $\alpha \in (0, \pi]$.

In the kernels of integral equations (2.3), the main parts corresponding to the inclusion in an infinite space (Problem A for the case when $\alpha = \pi$ when $K_{mn}(x, y, r, z) \equiv 0$ $(m, n = 1, 2)$; see [3], Eq. (1.16)) are separated out using formulae (1.8).

The dimensionless parameter λ introduced in (2.2) characterizes the relative remoteness of the inclusion from an edge of the wedge; the functions $K_{mn}(x, y, r, z) \to 0$ when $\lambda \to \infty$ $(m, n = 1, 2)$. To solve the system of integral equations (2.3), we use the regular asymptotic "large λ" method [7, 1–3], taking the exact solution of the problem of an inclusion in a space ([3], formulae (1.18)) as the zeroth approximation. The method is based on the following lemma.

Lemma 1. In the case of Problem A and B of an inclusion, the functions $K_{mn}(x, y, r, z)$ $(m, n = 1, 2)$ are continuous together with all their derivatives when (x, y), $(r, z) \in \Omega$. When

$$
\begin{aligned}
&\lambda > 1 + c \quad (\alpha \in [1,\pi]), \quad \lambda > \alpha^{-1} + c \quad (\alpha \in [c/2, 1]) \\
&\lambda > (1 + c^2(1+\alpha^2))^{1/2}\alpha^{-1} \quad (\alpha \in (0, c/2])
\end{aligned}
\tag{2.7}
$$

the functions $K_{mn}(x, y, r, z)$, (x, y), $(r, z) \in \Omega$ can be represented by the absolutely convergent series

$$
K_{mn}(x,y,r,z) = \sum_{l=1}^{\infty} \frac{k_l^{mn}(x,y,r,z)}{\lambda^{l+|m-n|}}, \quad m,n = 1,2 \tag{2.8}
$$

where $k_l^{mn}(x, y, r, z)$ are certain polynomials.

Well-known results ([1], the first two formulae of (2.5)) are used to prove the convergence. To obtain expansion (2.8), it is necessary to expand the trigonometric functions in formulae (2.4) in series. Integrals of the form

$$
\begin{aligned}
&\int_0^\infty \mathbf{K}_{iu}(\beta[\lambda + x])\mathbf{K}_{iu}(\beta[\lambda + r])\beta^{2j}d\beta = \\
&= \frac{(2j)!}{\lambda^{2j+1}}\int_0^\infty\int_0^\infty \frac{\cos(us)\cos(ut)dsdt}{[(1+x/\lambda)\mathrm{ch}\,s + (1+r/\lambda)\mathrm{ch}\,t]^{2j+1}}, \quad j = 0,1,\ldots
\end{aligned}
\tag{2.9}
$$

arise here, where the integral representation of a modified Bessel is used. The double integral (2.9) is expanded in a double Taylor series in powers of x/λ and r/λ. A formula ([11], formula 2.16.33.2), which can still be written in another form ([1], the last formula of (2.5)) is used to evaluate the integral coefficients in these expansions.

As a result, we obtain for the functions K_{mn} when $n = m$

$$
\begin{aligned}
K_{mm}(x,y,r,z) = {} & \frac{C_0^{mm}}{\lambda} + \frac{C_1^{mm}x + C_2^{mm}r}{\lambda^2} + \\
& + \frac{C_3^{mm}x^2 + C_4^{mm}xr + C_5^{mm}r^2 + C_6^{mm}(z-y)^2}{\lambda^3} + O\left(\frac{1}{\lambda^4}\right), \quad m = 1,2
\end{aligned}
\tag{2.10}
$$

where

$$
\begin{aligned}
C_0^{11} &= -\frac{1}{2}a_{10} + \frac{7-8\nu}{2}a_{20} - \frac{7-8\nu}{4}a_{30} - \frac{1}{4}a_{31} \\
C_1^{11} &= -\frac{1}{4}a_{10} + \frac{1}{8}a_{11} - \frac{7-8\nu}{4}a_{20} - \frac{1}{8}a_{21} + \frac{1-8\nu}{8}a_{30} - \frac{11-8\nu}{16}a_{31} \\
C_2^{11} &= \frac{3}{8}a_{10} - \frac{1}{8}a_{11} - \frac{7-8\nu}{4}a_{20} + \frac{1}{8}a_{21} + \frac{1-8\nu}{8}a_{30} + \frac{1-8\nu}{16}a_{31} \\
C_3^{11} &= \frac{1}{16}a_{10} + \frac{1}{32}a_{11} + \frac{3(7-8\nu)}{16}a_{20} - \\
&- \frac{15-8\nu}{32}a_{21} - \frac{3(7-8\nu)}{32}a_{30} + \frac{9-10\nu}{16}a_{31} + \frac{3}{256}a_{32} \\
C_4^{11} &= \frac{3}{8}a_{10} - \frac{5}{16}a_{11} + \frac{7-8\nu}{8}a_{20} + \\
&+ \frac{11-8\nu}{16}a_{21} - \frac{7-8\nu}{16}a_{30} - \frac{2-\nu}{4}a_{31} - \frac{3}{128}a_{32} \\
C_5^{11} &= -\frac{15}{16}a_{10} + \frac{9}{32}a_{11} + \frac{3(7-8\nu)}{16}a_{20} - \\
&- \frac{7-8\nu}{32}a_{21} - \frac{3(13-16\nu)}{32}a_{30} - \frac{3(1-2\nu)}{16}a_{31} + \frac{3}{256}a_{32} \\
C_6^{11} &= \frac{3}{32}a_{11} - \frac{9-8\nu}{32}a_{21} + \frac{19-24\nu}{64}a_{31} + \frac{1}{256}a_{32} \\
C_0^{22} &= \frac{7-8\nu}{2}a_{10} - \frac{1}{8}a_{31} + \frac{1}{2}a_{40} - a_{50} \\
C_1^{22} &= -\frac{7-8\nu}{4}a_{10} - \frac{1}{8}a_{11} + \frac{1}{16}a_{31} - \frac{3}{4}a_{40} + \frac{1}{8}a_{41} + \frac{3}{4}a_{50} \\
C_2^{22} &= -\frac{7-8\nu}{4}a_{10} + \frac{1}{8}a_{11} + \frac{1}{16}a_{31} + \frac{1}{4}a_{40} - \frac{1}{8}a_{41} - \frac{1}{2}a_{50} \\
C_3^{22} &= \frac{3(7-8\nu)}{16}a_{10} - \frac{3-16\nu}{32}a_{11} - \frac{5}{32}a_{30} - \frac{3}{64}a_{31} + \\
&+ \frac{1}{256}a_{32} + \frac{15}{16}a_{40} - \frac{9}{32}a_{41} - \frac{15}{8}a_{50} + \frac{1}{16}a_{51} \\
C_4^{22} &= \frac{9-8\nu}{16}a_{10} + \frac{9-8\nu}{32}a_{11} + \frac{5}{32}a_{30} + \frac{7}{64}a_{31} - \\
&- \frac{1}{256}a_{32} - \frac{9}{16}a_{40} + \frac{7}{32}a_{41} + \frac{7}{8}a_{50} - \frac{1}{16}a_{51} \\
C_5^{22} &= \frac{3(7-8\nu)}{16}a_{10} - \frac{11-8\nu}{32}a_{11} - \frac{5}{32}a_{30} - \frac{3}{64}a_{31} + \\
&+ \frac{1}{256}a_{32} - \frac{1}{16}a_{40} - \frac{1}{32}a_{41} - \frac{1}{8}a_{50} + \frac{1}{32}a_{51} \\
C_6^{22} &= -\frac{5-8\nu}{32}a_{11} + \frac{1}{256}a_{32} - \frac{3}{32}a_{41} + \frac{1}{16}a_{51}
\end{aligned}
\tag{2.11}
$$

Here, the notation

$$
a_{mn} = \frac{1}{4(1-\nu)}\int_0^\infty \mathrm{th}(\pi u) W_m(\alpha, u) f_n(u)\,du \tag{2.12}
$$

$$
f_0(u) \equiv 1, \quad f_1(u) = 1 + 4u^2, \quad f_2(u) = \left(1+4u^2\right)\left(9+4u^2\right)
$$

has been introduced, and, for the functions K_{mn} when $n \neq m$, we obtain

$$K_{mn}(x,y,r,z)=\frac{C_0^{mn}}{\lambda^2}(z-y)+\frac{C_1^{mn}x+C_2^{mn}r}{\lambda^3}(z-y)+O\left(\frac{1}{\lambda^4}\right),\quad m,\ n=1,2 \tag{2.13}$$

where

$$\begin{aligned}
C_0^{12} &= -\frac{7-8\nu}{4}a_{10}+\frac{1}{8}a_{11}+\frac{9-8\nu}{16}a_{31}-\frac{7-8\nu}{4}a_{40}-\frac{1}{8}a_{41}+\frac{7-8\nu}{8}a_{50}\\
C_1^{12} &= \frac{3-4\nu}{4}a_{10}+\frac{3-4\nu}{8}a_{11}+\frac{5}{32}a_{30}-\frac{9-16\nu}{64}a_{31}-\frac{1}{256}a_{32}+\\
&+\frac{3(7-8\nu)}{8}a_{40}-\frac{3-8\nu}{16}a_{41}-\frac{3(7-8\nu)}{4}a_{50}-\frac{1}{8}a_{51}\\
C_2^{12} &= \frac{3(7-8\nu)}{8}a_{10}-\frac{20-17\nu}{4}a_{11}-\frac{5}{16}a_{30}-\frac{3(9-8\nu)}{32}a_{31}+\\
&+\frac{1}{128}a_{32}+(1-\nu)a_{40}+\frac{1-\nu}{2}a_{41}-\frac{7-8\nu}{4}a_{50}+\frac{1}{8}a_{51}\\
C_0^{21} &= -\frac{1}{8}a_{11}+\frac{1}{8}a_{21}+\frac{3}{16}a_{31}\\
C_1^{21} &= \frac{1}{16}a_{11}-\frac{3}{16}a_{21}+\frac{5}{16}a_{30}+\frac{9}{32}a_{31}-\frac{1}{128}a_{32}\\
C_2^{21} &= \frac{3}{16}a_{11}-\frac{1}{16}a_{21}-\frac{5}{32}a_{30}-\frac{3}{64}a_{31}+\frac{1}{256}a_{32}
\end{aligned} \tag{2.14}$$

The values of the constants a_{mn} of the form (2.12) for $\nu = 0.3$ and different wedge angles 2α are shown in Table 1 for Problem A, when $a_{3k} \equiv 0$, $a_{4k} \equiv a_{2k}$, and for Problem B.

Now, on finding the solution of system (2.3), taking account of expressions (2.1) and (2.13) in the formula

Table 1

2α	$\pi/3$	$\pi/2$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
			Problem A			
a_{10}	1.305	0.6836	0.4124	0.1786	0.08062	0.02984
a_{11}	7.171	2.377	1.100	0.3571	0.1392	0.04728
$-a_{20}$	0.02763	0.1786	0.2062	0.1786	0.1256	0.06586
$-a_{21}$	-1.950	0.3571	0.5498	0.3571	0.1833	0.07207
a_{50}	0.6388	0.2525	0.1031	0	-0.02247	-0.01801
a_{51}	13.68	3.030	0.8246	0	-0.06611	-0.03719
			Problem B			
$-a_{10}$	0.4805	0.3265	0.2511	0.1786	0.1445	0.1255
$-a_{11}$	4.972	1.663	0.8213	0.3571	0.2292	0.1774
$-a_{20}$	0.5209	0.3708	0.2887	0.1786	0.1391	0.1249
$-a_{21}$	5.360	1.874	0.9328	0.3571	0.2204	0.1765
a_{30}	0.07561	0.1068	0.1354	0.1984	0.04305	0.007350
a_{31}	1.283	0.8428	0.6201	0.3968	0.09312	0.01453
a_{32}	72.52	24.18	11.75	4.762	1.054	0.1552
$-a_{40}$	0.3848	0.1786	0.04495	-0.1786	0.06166	0.1116
$-a_{41}$	3.050	0.3571	-0.1833	-0.3571	0.05282	0.1503
$-a_{50}$	0.9497	0.3532	0.1442	0	-0.01641	-0.006663
$-a_{51}$	46.66	7.492	1.659	0	-0.06285	-0.02083

$$\tau_n(x,y)=\sum_{l=0}^{\infty}\frac{\tau_{nl}(x,y)}{\lambda^l},\quad n=1,2 \tag{2.15}$$

and equating terms of like powers of the parameter λ, we obtain a sequence of systems of two integral equations for the successive determination of the functions $\tau_{nl}(x, y)$. Each system has the form

$$\int_{\Omega}\left[\tau_{1l}(x,y)R_{n1}+\tau_{2l}(x,y)R_{n2}\right]d\Omega_{xy}=F_{nl}(r,z)\quad (r,z)\in\Omega_{rz} \tag{2.16}$$

Here, $n = 1, 2$; $l = 0, 1, \ldots$ and $F_{nl}(r, z)$ are known polynomials.

The solution of system of equations (2.16) for a fixed value of l is based on the following lemma (see [3], p. 10).

Lemma 2. If the highest degree of the two polynomials $F_{1l}(r, z)$, $F_{2l}(r, z)$, for a fixed value of l, is equal to j, then the solution of system (2.16) can be represented in the form

$$\tau_{nl}(r,z)=\frac{Q_{nl}(r,z)}{L(r,z)},\quad L(r,z)=\left(1-\frac{r^2}{c^2}-z^2\right)^{1/2},\quad n=1,2 \tag{2.17}$$

where $Q_{nl}(r, z)$ are polynomials of degree j.

Note that $F_{10}(r, z) = 2\pi\delta$, $F_{20}(r, z) \equiv 0$ and the coefficients of the subsequent polynomials $F_{nl}(r, z)$ ($n = 1, 2$; $l = 1, 2, \ldots$) for each fixed value of l are integrals which now contain the functions $\tau_{nm}(r, z)$ ($n = 1, 2$; $m = 0, \ldots, l-1$), which are defined using Lemma 2 and are calculated using a well-known formula ([7], p. 45, formula (6)).

In order to determine the unknown coefficients of the polynomials $Q_{nl}(r, z)$ in formula (2.17), it is necessary to use previous results ([7], p. 44, formula (4) and [3], formula (1.7)). Finally, neglecting terms of the other of λ^{-4}, we obtain

$$\begin{aligned}\tau_1(r,z)&=\frac{\delta}{cL(r,z)}\left[T_{10}+\frac{T_{11}}{\lambda}+\frac{T_{12}+T_{13}r}{\lambda^2}+\right.\\&\left.+\frac{T_{14}+T_{15}r+T_{16}r^2+T_{17}z^2}{\lambda^3}+O\left(\frac{1}{\lambda^4}\right)\right]\\\tau_2(r,z)&=\frac{\delta}{cL(r,z)}\left[\frac{T_{20}z}{\lambda^2}+\frac{(T_{21}+T_{22}r)z}{\lambda^3}+O\left(\frac{1}{\lambda^4}\right)\right]\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}&T_{10}=\frac{T_{11}}{D_1}=\frac{T_{12}}{D_1^2}=\frac{1}{D_0},\quad D_0=S_{00}-\frac{S_{10}}{4(1-\nu)},\quad D_1=-\frac{C_0^{11}}{D_0}\\&T_{13}=\frac{T_{15}}{D_1}=\frac{4(1-\nu)}{D_0D_2}\left\{-C_0^{21}\left(S_{10}-3c^2S_{11}\right)-C_2^{11}\left[4(1-\nu)S_{10}-3c^2S_{11}\right]\right\}\\&T_{20}=\frac{T_{21}}{D_1}=\frac{4(1-\nu)c^2}{D_0D_2}\left\{-C_2^{11}\left(S_{01}-3S_{11}\right)-C_0^{21}\left[4(1-\nu)S_{01}-3S_{11}\right]\right\}\\&D_2=kc^2\left[(5-4\nu)S_{01}S_{10}-3\left(c^2S_{01}+S_{10}\right)S_{11}\right]\\&T_{14}=\frac{1}{D_0}\left[D_1^3-\frac{C_3^{11}c^2+C_6^{11}}{3D_0}-T_{16}\frac{c^2}{2}\left(S_{10}+\frac{2c^2S_{11}-S_{20}}{4(1-\nu)}\right)-\right.\\&\left.-T_{17}\frac{c^2}{2}\left(S_{01}-\frac{3S_{11}}{4(1-\nu)}\right)-T_{22}\frac{3c^2S_{11}}{8(1-\nu)}\right]\end{aligned} \tag{2.19}$$

The constants T_{16}, T_{17} and T_{22} are found from a system of three linear algebraic equations of the form

$$\begin{Vmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{Vmatrix} \begin{Vmatrix} T_{16} \\ T_{17} \\ T_{22} \end{Vmatrix} = \begin{Vmatrix} E_{14} \\ E_{24} \\ E_{34} \end{Vmatrix}$$

$$E_{11} = c^2\left[4(1-\nu)\left(2c^2 S_{02} - S_{11}\right) - 3\left(4c^2 S_{12} - S_{21}\right)\right]$$

$$E_{12} = -4(1-\nu)\left(c^2 S_{02} - 2S_{11}\right) + 3\left(3c^2 S_{12} - 2S_{21}\right)$$

$$E_{13} = 3\left[-c^2(2S_{11} + 3S_{12}) + 2S_{21}\right], \quad E_{14} = -8(1-\nu)C_5^{11} / D_0 \tag{2.20}$$

$$E_{21} = c^2\left[4c^2(S_{11} - 3S_{12}) + 4(1-\nu)\left(2c^2 S_{11} - S_{20}\right) - 2S_{20} + 3S_{30}\right]$$

$$E_{22} = -4(1-\nu)\left(c^2 S_{11} - 2S_{20}\right) - c^2(2S_{11} - 9S_{21}) + 2(2S_{20} - 3S_{30})$$

$$E_{23} = -3\left(3c^2 S_{21} - 2S_{30}\right), \quad E_{24} = -8(1-\nu)C_6^{11} / D_0$$

$$E_{31} = c^2\left[-2c^2(S_{02} - 6S_{12}) + S_{11} - 3S_{21}\right], \quad E_{32} = c^2(S_{02} - 9S_{12}) - 2(S_{11} - 3S_{21})$$

$$E_{33} = 3c^2\left[4(1-\nu)S_{11} + S_{11} - 4c^2 S_{12} + S_{21}\right], \quad E_{34} = -4(1-\nu)C_2^{21} / D_0$$

The notation

$$S_{mn} = \int_0^{\pi/2} \frac{\cos^{2m}\psi \sin^{2n}\psi}{\left(1 - e^2 \sin^2\psi\right)^{m+n+1/2}} d\psi, \quad e^2 = 1 - c^2 \tag{2.21}$$

has been introduced in relations (2.18) and (2.19).

The integrals (2.21) can be expressed in terms of the complete elliptic integrals $\mathbf{K}(e)$ and $\mathbf{E}(e)$

$$S_{00} = \mathbf{K}(e), \quad S_{01} = \frac{\mathbf{E}(e) - \left(1 - e^2\right)\mathbf{K}(e)}{e^2\left(1 - e^2\right)}$$

$$S_{02} = \frac{-2\left(1 - 2e^2\right)\mathbf{E}(e) + \left(2 - 5e^2 + 3e^4\right)\mathbf{K}(e)}{3e^4\left(1 - e^2\right)^2}$$

$$S_{10} = \frac{\mathbf{K}(e) - \mathbf{E}(e)}{e^2}, \quad S_{11} = \frac{\left(2 - e^2\right)\mathbf{E}(e) - 2\left(1 - e^2\right)\mathbf{K}(e)}{3e^4\left(1 - e^2\right)}$$

$$S_{12} = \frac{-\left(8 - 13e^2 + 3e^4\right)\mathbf{E}(e) + \left(8 - 17e^2 + 9e^4\right)\mathbf{K}(e)}{15e^6\left(1 - e^2\right)^2} \tag{2.22}$$

$$S_{20} = \frac{-2\left(1 + e^2\right)\mathbf{E}(e) + \left(2 + e^2\right)\mathbf{K}(e)}{3e^4}$$

$$S_{21} = \frac{\left(8 - 3e^2 - 2e^4\right)\mathbf{E}(e) - \left(8 - 7e^2 - e^4\right)\mathbf{K}(e)}{15e^6\left(1 - e^2\right)}$$

$$S_{30} = \frac{-\left(8 + 7e^2 + 8e^4\right)\mathbf{E}(e) + \left(8 + 3e^2 + 4e^4\right)\mathbf{K}(e)}{15e^6}$$

Table 2

λ	4			6			∞		
c	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
	Problem A								
T/δ	4.95	8.28	10.8	4.83	7.95	10.3	4.61	7.37	9.30
$f+/\delta$	2.77	0.915	0.651	2.71	0.888	0.631	2.59	0.829	0.581
$f-/\delta$	2.79	0.945	0.693	2.72	0.901	0.649	2.59	0.829	0.581
f/δ	2.49	1.87	2.00	1.08	0.802	0.853	0	0	0
	Problem B								
T/δ	5.07	8.63	11.4	4.91	8.17	10.6	4.61	7.37	9.30
$f+/\delta$	2.86	0.973	0.714	2.76	0.920	0.664	2.59	0.829	0.581
$f-/\delta$	2.86	0.976	0.719	2.76	0.921	0.666	2.59	0.829	0.581
f/δ	1.38	1.10	1.24	0.597	0.469	0.521	0	0	0

The relations between the force T, applied to the inclusion, and its displacement δ is found from the equilibrium condition

$$\int_{\Omega} \tau_1(x, y) d\Omega_{xy} = \frac{T}{2} \tag{2.23}$$

which, in the case of solution (2.18), takes the form

$$T = 4\pi\delta\left[T_{10} + \frac{T_{11}}{\lambda} + \frac{T_{12}}{\lambda^2} + \frac{1}{\lambda^3}\left(T_{14} + \frac{c^2}{3}T_{16} + \frac{T_{17}}{3}\right) + O\left(\frac{1}{\lambda^4}\right)\right] \tag{2.24}$$

The values of the quantities $T, f_\pm = \lim(1 \mp r/c)^{1/2}\tau_1(r, 0), r \to \pm c \mp 0$ and $f = 10^2 \cdot \lim(1-z)^{1/2}\tau_2(0, z)$, $z \to 1-0$, relative to δ, calculated using formulae (2.18)–(2.24) for $2\alpha = \pi/2$, $\nu = 0.3$ and different values of λ and c are given in Table 2

Note that Problem A for a wedge with an apex angle $2\pi/n$ ($n = 1, 2, \ldots$) corresponds to the symmetrical problem of n identical inclusions in an infinite space; the inclusions are arranged in half-spaces, the angle between which is $2\pi/n$. The interaction of four inclusions in a space is investigated for a quarter space (Table 2, Problem A). The closer these inclusions are to one another (the smaller λ), the greater the force T required to displace the inclusions by a specified amount δ. In the case of Problem B, the corresponding value of T is greater than in the case of Problem A. A circular inclusion ($c \to 1$) is more difficult to move than one having a large eccentricity (the value of c is small). The coefficient of the root singularity of the shear contact stress $\tau_{r\varphi}$ is somewhat greater on the side of the inclusion which is closer to the edge of the wedge ($f_- > f_+$). The estimate $\tau_{\varphi z} = O(\tau_{r\varphi}/\lambda^2)$, $\lambda \to \infty$ holds in the case of the shear of an inclusion perpendicular to the edge for transverse motion of the shear contact stress $\tau_{\varphi z}$.

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